Research in Applied Mathematics in the UK

The definition of Applied Mathematics employed in the Research Assessment Exercise reads as follows:

> *Applied Mathematics consists of the development of, the analysis of, and the solution or approximate solution of, mathematical models including those arising in physical and biological sciences, engineering and technology, and the development and application of mathematical theories and techniques that further these objectives. There are therefore overlaps between Applied Mathematics and (a) the application areas and (b) other branches of mathematics and physics, including theoretical physics and, in the context of numerical or symbolic calculation, (c) with computer science. The unit also includes pedagogic research in Applied Mathematics*

This definition provides a wide-ranging view of the nature of Applied Mathematics. Traditionally, the greatest strength of British Applied Mathematics has been widely perceived to lie in theoretical mechanics (fluids and solids, but especially fluids) and its applications. The type of mathematics involved most often requires the analytical, asymptotic or computational solution of partial differential equations (PDEs), for example the Navier-Stokes equations. It has therefore been natural for British applied mathematicians to move into areas that are not strictly speaking mechanics but in which appropriate mathematical modelling also results in partial differential equations. Examples include several areas of mathematical biology, mathematical chemistry, mathematical finance, etc. Of course, once in those areas, applied mathematicians discover new types of problem for which the best mathematical models are not PDEs -- for example, because the natural formulation involves discrete not continuous variables or because there are memory or hysteresis effects, or because stochastic events are significant so that probabilistic formulations and methods are required. In addition, there are new areas of theoretical physics research that lie between classical mechanics on the one hand and microscopic, quantum mechanics on the other (e.g. polymers, liquid crystals, wetting phenomena, liquid helium flow, etc) and British applied mathematicians have not been slow to move into such subjects.

In fundamental theoretical fluid mechanics there are major research groups of international quality in boundary layer theory, convection, hydrodynamic stability, thin-film flow with free surfaces of constant or variable surface tension, water waves, low-Reynolds-number flows, vortex dynamics, among others. There exist a small number of good researchers applying rigorous mathematics to fluid dynamics but, except in topological fluid dynamics, this is not as well developed as in some other countries. A significant weakness in British Applied Mathematics is the small number of people in fundamental turbulence theory.

On the other hand there is a wealth of high quality talent studying more applied problems. A major field is that of multiphase flows (bubbles, singly and in clouds, suspensions of neutrally buoyant or dense particles or drops, and particle laden gases, with or without phase change), with applications in the oil industry, in studies of environmental pollution and in geophysics (e.g. magma chambers, volcanic gas clouds, turbidity currents, avalanches). Another major area of application, in several institutions, is the dynamics of atmosphere and oceans, both leading to fundamental understanding of the flow of stratified and/or rotating fluids and hence of dynamical processes on all scales in meteorology and oceanography, and contributing to programmes of detailed computational simulation of air and sea behaviour at particular locations on the earth's surface. With an overlapping interest in rotating and/or stratified fluids the number of institutions (eleven) submitting work in magnetohydrodynamics (MHD) has grown. The primary areas of application are in astrophysics and geophysics with some work of industrial relevance. Numerical simulation is increasingly important as computational power advances, permitting more realistic problems to be tackled. In some areas, major relevant observational programmes have stimulated significant activity in the analysis of the data. Closely related MHD work takes place in departments submitted to UoA19 and 20.

Because the equations of fluid dynamics are highly non-linear, virtually every new application brings new opportunities for modelling and for learning new physics by mathematical means. Thus many researchers in most substantial Applied Mathematics departments are involved in interdisciplinary projects of some sort, with applications (particularly) in industry or in medicine and biology. Among the industries to which British applied mathematicians are contributing are aerospace, oil, glass, textiles, food, fuel cells, liquid crystal displays, printing, and the many applications in which combustion is important, a sub-area of fluid dynamics in which there is strength at two or three institutions. In medicine there are applications to physiology (blood flow, lungs, urodynamics), neo-natal care, understanding and treating cardiovascular disease, and the growth of tumours. New developments are focussing on mechanics at the cellular level. In non-medical biological fluid dynamics there are only a few individuals, working mainly on animal locomotion – of fish, insects, microorganisms - though with applications to ecology.

Studies of tumour growth, for example, employ systems of reaction-diffusion equations in which the reaction terms reflect population growth (of cells) as well as the uptake and discharge of chemicals. Many biological processes can also be described by such systems of equations - wound-healing, the excitation of heart muscle, embryo development, plankton-population dynamics (and other predator-prey systems), to name only a few that are studied in the UK. British applied mathematicians have been and remain at the forefront of the analysis of reaction-diffusion equations, leading to a profound understanding of pattern-formation, chemical wave propagation, excitable media, non-linear diffusion, etc. Moreover, the equations of population dynamics, epidemic modelling and evolution are similar, and there are several strong groups studying them. Aspects of these areas make it natural to study discrete and/or stochastic systems, and there are some groups with a different mathematical background (game theory, probability theory, etc) making significant progress. Also, there is appreciable emphasis on the study of nerve impulses and their non-linear dynamics.

Research submitted to UoA23 in solid mechanics and general continuum mechanics covers a number of themes. There are few institutions containing sizeable groups in solid mechanics, but inter-institutional collaborations are evident.

In finite elasticity, besides development of general techniques for incompressible and near-incompressible materials, for statics, dynamics and bifurcation from uniform states, there are examples of bio-medical applications and of analysis of composites and structured materials. A novel feature is an emphasis on describing surface-coating effects. There have been substantial advances in the theory of crack propagation and fracture, largely by using linear theory but also by using non-linearity or viscoelasticity. Three-dimensional and dynamic effects are being tackled, as are practical problems including the advance of cracks along interfaces between dissimilar materials. Activity in plasticity, powder mechanics and granular flow is limited, though attempts to resolve long-standing difficulties with the formulation are evident. Related ideas appear also in models of large-scale geophysical phenomena, such as pack ice and glacial flow.

In a number of institutions, there is activity in describing multi-phase materials having fine structure at the microscopic level. The work utilizes 'weak solutions' and is analytically intricate. Similar techniques are more advanced for describing homogenised thermal and electric properties. Also, a number of groups are developing descriptions of superconductivity, where again weak solutions are utilized. Liquid crystals are tackled in a number of strong groups, from various viewpoints. These range from (macro-) molecular theories using statistical physics to continuum theories, for which solutions to boundary value problems appropriate to practical device geometries.

In various aspects of acoustics, a number of strong inter-institutional collaborations are evident, though few institutions have substantial acoustic groups. U.K. teams continue to make significant advances in developing Wiener-Hopf techniques for increasingly complex geometries. Strong international links exist, particularly with Russia. Many of the problems tackled are canonical ingredients in larger problems and are incorporated into numerical schemes for tackling those. Examples are the scattering from edges, crack tips and tips of cones. Particular activity in exploiting the Maliuzhinets' function is evident. In the U.K. acoustic theory has long been applied in aeronautics and in marine technology. Now, there is considerable emphasis on environmental aspects of traffic noise (both road and rail).

In acousto-elasticity, Green function techniques are being refined for periodic arrays of inclusions and transfer-matrix and modal analysis methods applied to analysis waves in laminated plates and cylindrical shells. Techniques for non-linear elastic waves have strong similarity with wave train modulation theories arising in fluids and elsewhere in mathematical physics.

Electromagnetic waves are being analysed both in practical problems of radar systems and industrial wave-guides and as intricate mathematical scattering problems giving rise, for example, to exponential asymptotics. Non-linear electromagnetism is studied in the U.K. in the context of laser dynamics and non-linear optics, the latter driving numerical, perturbative and exact advances for non-integrable systems of PDEs. This work interacts with aspects of non-linear analysis.

Industrial mathematics is a major emphasis of some institutions. It includes a number of items listed already and also others such as semiconductor design and manufacture, oxidation and its interaction with fracture, and electro-rheological fluids and devices, where mathematical analysis is driven by industrial issues. This research is in many instances supported by industrial funding. Additionally, many groups commit considerable effort to the Mathematical Study Groups with Industry, an initiative which started in Britain and which now takes place in many countries. Since the aim of Study Groups is to address practical industrial problems, problems do not always generate publishable output. Consequently, few of the RA2 outputs reflect the major commitment of time and intellectual effort of some academics. That commitment is however reflected by a large number of invitations to lead activities abroad. Some trends in U.K. research activity (e.g. semiconductor fabrication) are historically traceable to Study Group problems. Mathematical Study Groups in Medicine have also been started recently.

Thus the aim of much of British Applied Mathematics is to understand phenomena from outside mathematics, often through interdisciplinary collaboration. A number of institutions group research topics under the title of engineering mathematics. Many examples of such interdisciplinary topics have been mentioned above, but others include traffic flow, statistical analysis of accidents, industrial reliability, vibrations of suspension bridges and use and development of CFD software. In some instances, applied mathematicians rightly become so immersed in the collaboration that the mathematical content of their work is not the dominant feature. In these circumstances, their work must be judged by its contribution to science, not narrowly as mathematics. Examples include work on earth observation, remote sensing, sedimentary analysis, weather measurement, drug delivery and laser drilling.

By its nature the subject of Applied Mathematics overlaps with all the different sciences to which mathematics can be applied, the oldest and most mathematical of them being physics. The position of the boundary is a matter of tradition and convention, and differs in different countries. In many countries a particular arbitrary selection of areas of physics is regarded as belonging to Applied Mathematics, while other areas, despite being highly mathematical (indeed, closer in many ways to pure mathematics than what is regarded as Applied Mathematics) tend to be studied exclusively in Physics departments of universities. But Britain, the country of Newton, Maxwell and Dirac, has a tradition of mathematics departments producing great work across the full range of mathematical and theoretical physics, all of which is covered by the term "Applied Mathematics". The present assessment exercise has shown that this tradition is living and very healthy; there are household names working with research groups whose work is likely to be of historically lasting value. About 20% of the research surveyed by UoA 23 is in areas of physics that might not be studied in Applied Mathematics departments in some other countries.

Throughout the world, research into the fundamental laws of physics, and on the fundamental problem of quantum gravity, is currently focussed on theories of strings and higher-dimensional objects in spaces of ten or eleven dimensions. Several of the world leaders in this area are based in the UK, with highly active groups in a number of institutions. The work surveyed in the assessment was particularly concentrated on the theory of branes, the AdS/CFT correspondence, the possibility of an overarching theory called M-theory, and the emergence of non-commutative geometry at small length scales as an essential ingredient in fundamental physical theory. Conformal field theory is studied more abstractly, both for its intrinsic mathematical interest and relation to other purely mathematical topics, and for its applications in other areas of physics. Supergravity and alternative approaches to the search for a theory of quantum gravity inspires an appreciable amount of research. The great majority of all of this work is internationally excellent.

There is a considerable amount of work, also largely of international excellence, in quantum field theory, including rigorous theory in curved space-time, studies of renormalization, new calculations of deep inelastic scattering, development of gauge theories and supersymmetric theories, and other topics. Supersymmetry is by itself the subject of a substantial amount of research. Solvable (or "integrable") models in quantum field theory are being studied intensively; together with classical integrable systems, this forms a major area. During the period of the assessment there was a growing interest in systems with integrable boundary conditions. There has also been significant numerical study of solitons, monopoles, skyrmions and similar solutions of field theories.

Applications of quantum field theory to elementary particle physics account for a small but significant part of the research assessed, particularly including large-scale lattice QCD computations in which particle properties are calculated from the standard model. At a lower energy scale, some applied mathematicians work on atomic and molecular physics and theoretical chemistry, with relativistic calculations now being embarked upon. There is a small amount of work on nuclear and solid state physics.

Purely theoretical work on quantum mechanics includes rigorous studies of the spectrum of Schro”dinger operators, the development of quantum probability theory, and a number of other approaches to the foundations of quantum mechanics. A recent development is the growth of the new subject of quantum information theory. Moving from the very small to the very large, there is a substantial amount of work on astronomy, astrophysics and cosmology. The theoretical basis for this is general relativity, which is a major subject in British Applied Mathematics. Topics studied include gravitational waves, black holes and other singularities, fluids, the classification of spacetimes, and exact solutions of Einstein's equations. There is a certain amount of numerical work. At the mathematical level, twistor theory continues to be developed, and links have been explored with the theory of integrable systems. Cosmology is a major area, attention being noticeably devoted to cosmic strings and structure formation.

Much of theoretical physics depends on ideas of symmetry, and the Applied Mathematics unit of assessment includes some purely mathematical work on groups, Lie algebras and quantum groups. This has led to some topological work on invariants of 3- and 4-manifolds. Other work arising from physics, which has applications in pure mathematics, is the possibly highly significant work on random matrices, with applications in number theory.

The scope of UoA23 includes Applied Analysis, a term used here to denote rigorous analysis of differential equations and related mathematics, arising from scientific problem areas. In many countries it is the principal component of Applied Mathematics. Six submissions presented a substantial amount of internationally excellent research activity in Applied Analysis. A further twenty-one submissions presented moderate activity in the area. The scientific problems treated range from control, elasticity, and bifurcations in fluids and lasers to integrable systems, quantum mechanical spectra and probabilistic problems. The remaining 31 submissions contained little or no Applied Analysis.

Numerical Analysis was returned to UoA23 by a relatively small number of universities. Some areas of numerical analysis are closely linked to applied analysis interpreted in a wide sense, others to application areas (such as system identification, viscoelasticity, and problems of control). In some cases, research in numerical analysis was returned to UoA 25 (Computer Science). A substantial proportion of submissions in numerical analysis display clear international quality. Several groups returned work of international excellence in the construction and analysis of algorithms for numerical solution of various types of differential equations. The outputs submitted show that significant contributions are being made in areas that include geometric integration, hp-finite element approximations, functional differential equations, long-time dynamics of numerical methods, stochastic differential equations of various types, adaptivity and error estimation. Work of international significance was also returned by researchers in numerical linear algebra, integral (and integro-differential) equations, optimisation, and approximation theory. There is a long-standing and powerful base in numerical analysis and computational fluid dynamics within the UK computational mathematics community. Many submissions show that there is recent growth in other areas of application such as phase boundary motion, superconductivity, non-Newtonian fluid flows and in the life sciences (e.g., immunology, physiology, pharmacy).

Any mention of the relative strengths of an area brings with it the suspicion that there are also relative weaknesses. The Panel for UoA 23 note that they received only one submission from Wales and none from Northern Ireland. It is worthwhile observing that many of the recent new appointments in Applied Mathematics are of staff who have come to the UK from educational systems in other countries. This can bring with it a diversity of talents and mathematical traditions that are a positive feature but may bring concerns for the indigenous young talent. There is also some concern to ensure that any barriers between various areas of the mathematical sciences (in particular between the areas of mathematics covered by UoAs 22, 23 and 24) should not be allowed to have an adverse effect on the development of the subject. The fact that over thirty submissions contained little or no Applied Analysis could be viewed as a weakness of the UK Applied Mathematics scene which may have been perpetuated by the separation of mathematics into pure, applied and statistics in many UK systems. There is scope for further development of the holistic approach to the subject whereby, for example, modelling, analysis, numerical analysis and computational science are integrated (in the case of interdisciplinary work, in conjunction with scientists in the application area). It is desirable that the criteria employed in Research Assessment Exercises should be seen to encourage such holistic approaches, where the value of the whole may be greater than the value of the component parts. Traditional modelling techniques allied with applied analysis, numerical analysis and computational modelling in application areas where either qualitatively correct or quantitatively correct models are required produce valuable scientific benefits and should be encouraged (in particular by funding agencies). With regard to sources of funding, there appear to be opportunities for applied mathematicians to promote the support of research by funding from an enlightened industrial base that is made more aware of the benefits available.

For further information on the state of Applied Mathematics in the UK, we refer to the annual reviews of the Mathematics Programme of EPSRC.

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(signed) C T H Baker on behalf of the RAE2001 Panel for UoA23

Chair

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